

Fractality of DLA in an arbitrary Euclidean dimension beyond the on-lattice and mean-field approximations

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The derivation of a consistent analytical description for the fractality of the paradigmatic diffusion-limited aggregation (DLA) model, valid beyond the on-lattice and mean-field approximations in any embedding space, has been lengthy pursuit issue in out-of-equilibrium growth research. Here, by unifying previous renormalization-group and mean-field proposals, we provide such general solution to the scaling of the DLA cluster and to its harmonic measure as well. This result is in excellent agreement with available and reliable results for the fractal dimensions of DLA reported over the years, and in particular, it consistently establishes $D = 1 + 1/\sqrt{2} \approx 1.707$, for two-dimensions.

INTRODUCTION

The diffusion-limited aggregation (DLA) model^{1,2} is a paradigm for out-of-equilibrium growth³⁻⁵, of great relevance in diverse scientific and technological fields, such as oil industry, bacterial growth and even cosmology⁶⁻¹⁰. Its characteristic structure has been the subject of extensive numerical¹¹⁻¹⁹ and theoretical²⁰⁻⁴⁶ research, that have revealed that this system can be characterized by a single fractal dimension, D , only dependent on the Euclidean dimension of its embedding space, d . On the basis of these extensive studies, it is presently accepted that $D = 1.71$ is the correct scaling for the off-lattice radial system in $d = 2$, whereas for higher dimensions, this consensus is not that clear, as it can be seen in Table I, where we present the results obtained for $D(d)$ through diverse numerical and theoretical approaches over the years. In addition, the derivation of a general and consistent analytical expression for $D(d)$ in the off-lattice scenario, valid not only for $d = 2$, but in higher dimensions as well, has proven to be a non-trivial task and has been missing^{5,7}. Among the diverse theoretical approaches to the problem, we will employ two of them as starting point of our analysis to derive this desired analytical expression.

First, under the mean-field (MF) approach^{26,27,32}, a close expression for $D(d)$ was found, as given by,

$$D(d) = \frac{d^2 + 1}{d + 1}. \quad (1)$$

Although this expression is valid in off-lattice conditions, it predicts $D = 5/3 \approx 1.67$, for $d = 2$, which is an underestimation of the known scaling. Also, it does not give a proper description of the reported data for higher dimensions, as seen in Fig. 1, where we compare this to other theoretical and numerical results. Second, under a renormalization-group (RG) approach^{36,37}, it was possible to find an expression for $D(d)$, given as,

$$D(k, d) = 1 + \frac{\log \mu(k, d)}{\log 2}, \quad (2)$$

with,

$$\mu(k, d) = 1 + 2 \binom{d-1}{1} \phi_1 + \sum_{k=2}^{d-1} \binom{d-1}{k} \phi_k, \quad (3)$$

where, $\phi_k(d)$ are the growth potentials for a given lattice, and $\mu(k, d)$ is inversely proportional to the probability of maximum growth, $\mu(k, d) = 1/P(d)_{max}$. From here, one may infer that the strong dependence of this model on the lattice, makes it inevitably susceptible to well known inherent lattice effects. In particular, for a square lattice in $d = 2$, it predicts a scaling of $D = 1.737$, which is an overestimation of the known value.

Based on these MF and RG results, in this work we develop a model that unifies both descriptions and provides a consistent and general expression for $D(d)$, valid off-lattice and in any spatial dimension d as well. All the data used in this work is presented in Table I.

THE MODEL

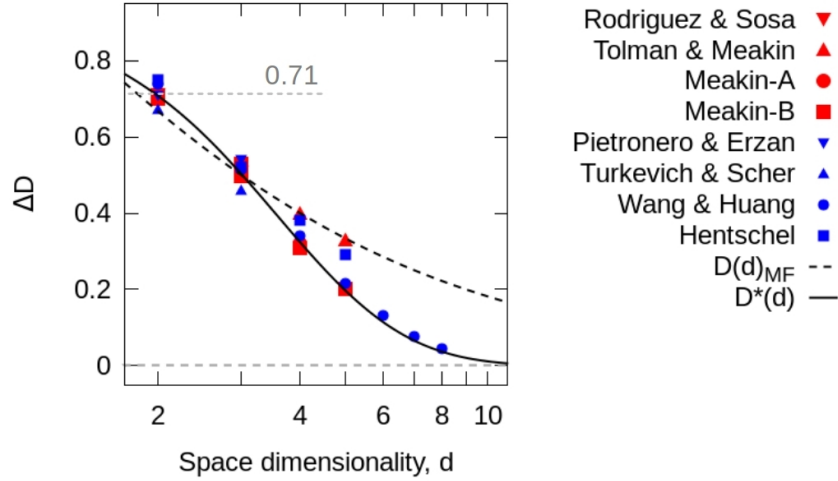


FIG. 1. **Fractal dimensions of DLA.** The fractal dimensions, $D(d)$, must always satisfy the Ball inequality given as, $D(d) - (d - 1) \geq 0$, where the equality is valid for $d \rightarrow \infty$ ²⁵. Since $D(d)$ is a monotonically increasing function of d , then $\Delta D = D(d) - (d - 1)$ is monotonically decreasing. Numerical (red) and theoretical (blue) results reported over the years satisfy this condition (references in Table I). However, the mean-field result, $D(d)_{MF}$, given by Eq. (1), is not sufficient to establish a precise description of the data. The curve for $D^*(d)$ is the solution derived in this work.

We start by introducing, $\Lambda(d)$, a spatial correction function for DLA. This quantity naturally appears from the MF equation and is crucial to validate any solution for $D(d)$. To construct it and appreciate its relevance, let us make evident that Eq. (1), can be recovered

from the first-order approximation in $f(d) = \Lambda(d)/d$, of a general exponential form given by,

$$D(d) = 1 + (d - 1)e^{-f(d)}. \quad (4)$$

From here, it easily follows that,

$$D(d) \approx 1 + \frac{d - 1}{1 + f(d)} = \frac{d + f(d)}{1 + f(d)} = \frac{d^2 + \Lambda(d)}{d + \Lambda(d)}, \quad (5)$$

then, by setting $\Lambda(d) = 1$, the MF equation is recovered. Under this description, one can observe that $f(d)$ is the term that keeps all the information associated to the symmetry-breaking of space in DLA. However, in the MF approximation, $\Lambda(d) = 1$, for all d , and in the case of $d = 2$, this predicts $D = 5/3 \approx 1.67$, which is not in the best agreement with results, even for higher dimensions (see Fig. 1 and Table I). For a better understanding of $\Lambda(d)$, let us define it through Eq. (4), by the variable transformation,

$$\Lambda(d) = -d \log(\hat{D}), \quad (6)$$

where $\hat{D} = (D - 1)/(d - 1)$, is the reduced co-dimension of DLA. By applying the previous transformation to all the available numerical results of fractal dimensions for DLA (Table I) and comparing to the MF model, $\Lambda(d) = 1$, in Fig. 2a, it is clear to see the non-trivial behavior of $\Lambda(d)$, and its important role as the spatial correction factor for DLA. Consequently, the problem of finding the correct scaling for DLA relies on finding a suitable $\Lambda(d)$.

Even though there can be diverse ways to do this, we did it by extending the on-lattice RG results, to be valid in the continuous space, as follows. From Eqs. (2) and (3), their extension to the continuous space, i.e., $\mu(k, d) \rightarrow \mu(d)$, is done, without loss of generality, by considering $\phi_k(d) = \Phi(d)$ for all k , where $\Phi(d)$ is now a continuous function that exclusively depends on d . This leads to,

$$\mu(d) = 1 + (2^{d-1} + d - 2)\Phi(d), \quad (7)$$

which substituted back into Eq. (2) and solving for $\Phi(d)$, gives,

$$\Phi(d) = \frac{2^{D(d)-1} - 1}{2^{d-1} + d - 2}. \quad (8)$$

From the known solutions of $\phi_k(d)$ in the square lattice, the ansatz for $\Phi(d)$ is expected to be close to $1/3$ as $d \rightarrow 2$, and to tend to $1/2$ as $d \rightarrow \infty$. In this particular limit, $\mu(d, \Phi \rightarrow 1/2) = 2^{d-1} + d/2$, is consistent with what it was previously found³⁷. Furthermore, $\Phi(d)$ must be such that P_{max} remains bounded to $[0, 1]$, and the Ball inequality, $D \geq d - 1$ (where the equality holds for $d \rightarrow \infty$) is satisfied for any d^{25} . In Fig. 2b, by applying Eq. (8) to all the available numerical results of fractal dimensions for DLA, we can observe the expected behavior of $\Phi(d)$.

Notice that finding a suitable ansatz for $\Phi(d)$ in this problem is equivalent to finding an independent solution for $D(d)$, through the extension of the RG equation to the continuous space. Therefore, what we finally obtain here, is a solution to our problem in which the extended RG description, Eq. (7), provides the correction factor through $\Lambda(d)$, to the MF expression itself, Eq. (4), needed to convey a consistent and unified description of $D(d)$.

Finally, since $\Lambda(d)$ depends directly on $D(d)$, it is possible to establish two restrictions for it. First, Eq. (4), along with the Ball inequality, $D \geq d - 1$, impose an upper boundary, $\Lambda^+ = -d \log[(d - 2)/(d - 1)]$. Second, in the $d \rightarrow \infty$ limit, $\mu(d, \Phi \rightarrow 1/2) = 2^{d-2} + d/2$, leads to a lower boundary, $\Lambda^- = -d \log[\log \mu(d) / \log 2^{d-1}]$. Thus, a solution for $D(d)$ must be such that the inequality, $\Lambda^- \leq \Lambda(d) \leq \Lambda^+$, should be always satisfied. In both cases, the equality will hold for $d \rightarrow \infty$. In fact, this condition is satisfied at $d \simeq 10$, when both MF and RG descriptions become equivalent (see Fig. 2a).

RESULTS

Under the previous considerations, we start the construction of the ansatz for $\Phi(d)$ by observing that when $d \rightarrow \infty$, all the information in $\mu(d)$ regarding $D(d)$ as $d \rightarrow 2$ is lost, as seen through Λ^- , in Fig. 2a. This information can be recovered by taking $\phi_k = 1/2$ (the limit-value of ϕ_k as $d \rightarrow \infty$) starting from $k \geq 2$. This leads to,

$$\mu(d) = 1 - d/2 + 2^{d-2} + 2(d - 1)\phi(d), \quad (9)$$

where $\phi_1 \rightarrow \phi(d)$, is a continuous function of d , and is expected to satisfy the previous considerations for $\Phi(d)$ at low dimensions. Also, since both Eqs. (7) and (9) are equivalent, their respective ansatz should obey,

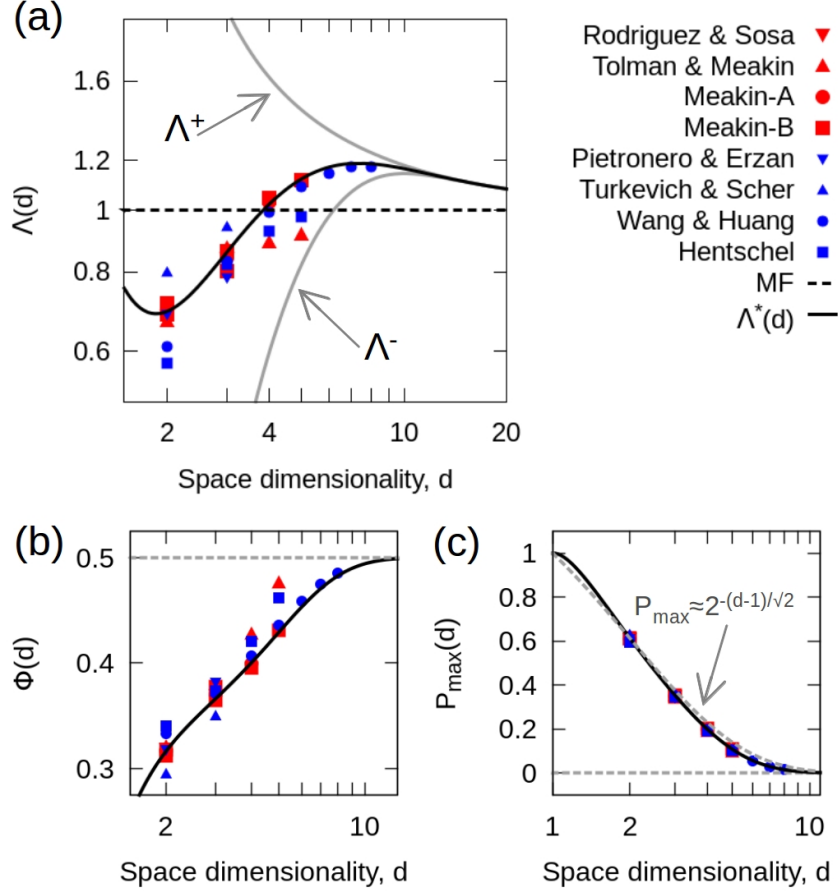


FIG. 2. **Criteria for a consistent solution of $D(d)$.** (a) The behavior of $\Lambda(d)$ as given by data, the boundaries Λ^+ and Λ^- , as well as the mean-field model $\Lambda(d) = 1$, reveal the non-trivial behavior of the solution for $\Lambda(d)$. The solid line is given by Eq. (13). (b) $\Phi(d)$ as given by data and by Eqs. (10) and (11). (c) $P_{\max}(d) = \mu(d)^{-1}$ as given by data and through Eq. (12) are shown, as well as the approximation for P_{\max} valid at low dimensions (dotted-line).

$$\Phi(d) = \frac{2^{d-2} - d/2 + 2(d-1)\phi(d)}{2^{d-1} + d - 2}. \quad (10)$$

Therefore, a solution for $\phi(d)$ guarantees a solution for $\Phi(d)$ as well. Now, as $d \rightarrow 2$, from Eqs. (8) and (10), we have that $\Phi(d) = \phi(d) = (2^{D(d \rightarrow 2)-1} - 1)/2$. Then, from Eq. (9), we have that $\mu(d \rightarrow 2) = 1 + 2\phi(d \rightarrow 2) = 2^{D(d \rightarrow 2)-1}$. This form for $\mu(d \rightarrow 2)$, along with Eq. (2), suggest that D can be approximated as, $D(d \rightarrow 2) - 1 \approx (d-1)/\sqrt{2}$, which for $d = 2$, gives $D = 1 + 1/\sqrt{2} \approx 1.707$, as previously found by other method³⁰. Notice that what we have here is an approximation to the probability function itself given as $\mu^{-1} = P_{\max} \approx 2^{-(d-1)/\sqrt{2}}$, which is valid as $d \rightarrow 2$, see Fig. 2c. Therefore, based on these

observations, we propose,

$$\phi(d) = \frac{2\sqrt{(d-1)/2} - 1}{d}, \quad (11)$$

as our ansatz to the problem. As it can be observed, this expression for $\phi(d)$ yields a great agreement to the data and complies with all the necessary conditions for it to be a unique and consistent solution to the problem. When plugged back into Eq. (9), this gives,

$$\mu(d) = 1 - \frac{d}{2} + 2^{d-2} + 2\left(\frac{d-1}{d}\right)[2\sqrt{(d-1)/2} - 1], \quad (12)$$

that provides a nice description for $P(d)_{max}$, Fig. 2c. By substituting back Eq. (11) into Eq. (10), it provides the expected behavior for $\Phi(d)$, Fig. 2b. Also, the solution for $\mu(d)$ can then be substituted back into Eq. (2) to obtain a continuous analytical function for $D(d)$, that at the same time, from Eq. (6), provides the solution for $\Lambda(d)$ itself, this is,

$$\Lambda(d) = -d \log[\log \mu(d) / \log 2^{d-1}], \quad (13)$$

that satisfies the restrictions imposed by the theory, including $D(d) \geq d - 1$, for all d (Fig. 2a). In particular, this solution gives $D = 1 + 1/\sqrt{2} \approx 1.707$ for $d = 2$, in good agreement with the scaling of DLA in two and also, higher dimensions (Fig. 1 and Table I). Furthermore, as previously stated, this expression for $\Lambda(d)$ allows for both MF, Eq. (4), and extended RG descriptions, Eqs. (2) and (12), to become nicely and consistently equivalent.

Finally, it must be pointed out that, since Eq. (2) is in fact the Turkevich-Scher relation^{29,37}, that relates the $D(d)$ of the structure with the scaling of its harmonic measure, $\alpha(d)_{min}$, as $D(d) = 1 + \alpha(d)_{min}$, the solution derived here for $D(d)$, given by the extended Eqs. (2) or (4), also provides the solution to the scaling of its harmonic measure. Even more, from Eq. (4), we can see that, $dD/df = d(D - 1)/df = -(D - 1)$, where $f(d) = \Lambda(d)/d$. Then, $\alpha_{min} = (d - 1) \exp[-f(d)]$ remarkably satisfies $d\alpha_{min}/df = -\alpha_{min}$. Hence, this differential equation clearly shows that α_{min} , through the weighted spatial correction function, $f(d)$, governs the spatial symmetry-breaking of the diffusion-limited growth process.

TABLE I. **DLA dimensions.** In the first section, numerical results for $D(d)$ of clusters grown in square lattices are presented, otherwise indicated. Results marked with (*) are off-lattice. Error in measurements shown when available. In the second section, theoretical and semi-numerical results are displayed, along with the results obtained in this work (bottom line).

(Year) Author(s)	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$	$d = 8$
(1983) Meakin-A ¹¹	1.71 ± 0.05	2.51 ± 0.06	3.32 ± 0.10				
(1983) Meakin-B ¹²	1.70 ± 0.06	2.53 ± 0.06	3.31 ± 0.10	4.20 ± 0.16	≈ 5.35		
	$1.71 \pm 0.07^*$	$2.50 \pm 0.08^*$					
(1989) Tolman & Meakin ¹³	$1.715 \pm 0.004^*$	2.495 ± 0.005	≈ 3.40	≈ 4.33	≈ 5.40	≈ 6.45	≈ 7.50
(2013) Rodriguez & Sosa ¹⁹	$1.711 \pm 0.008^*$	$2.51 \pm 0.01^*$					
(1984) Hentschel (MF) ²⁸	1.75	2.52	3.38	4.29	5.0		
(1985) Turkevich & Scher ²⁹	1.67	2.46					
(1992) Halsey ³⁸	1.66	2.50	3.40	4.33			
(1995) Erzan, <i>et al.</i> ⁴¹	1.71	2.54					
(1983-86) MF ^{26,27,32}	1.67	2.50	3.40	4.33	5.29	6.25	7.22
(1992) Wang & Huang (RG) ³⁷	1.737	2.515	3.341	4.216	5.131	6.077	7.045
From Eqs. (2) and (12)	1.707	2.503	3.324	4.196	5.114	6.065	7.037

CONCLUSIONS

Based on well established MF and RG results for the fractal dimension of DLA, we provide a solution to a long standing problem in fractal growth research, this is, the derivation of the first analytical expression for the fractal dimension of DLA in the continuous-space or off-lattice model. This MF+RG description for $D(d)$ is valid in the original two-dimensional version of the problem and satisfies the most rigorous conditions to have a consistent scaling for higher dimensions as well. Furthermore, it provides the necessary corrections to make both, MF and RG descriptions consistently equivalent. Finally, it provides a complete solution to the scaling of the harmonic measure, $\alpha(d)_{\min}$, while confirming its importance in defining the self-similar features of the DLA structures.

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